

Common Fixed Point Theorems for Multivalued Generalized F -Suzuki-Contraction Mappings in Complete Strong b -Metric Spaces

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Abstract

The aim of this paper is to introduce a new version of multivalued generalized F -Suzuki-Contraction mapping and then establish some new common fixed point theorems for these new multivalued generalized F -Suzuki-Contraction Mappings in complete strong b -metric Spaces.

Keywords: Common Fixed Point Problem; Multivalued Generalized F -Suzuki-Contraction Mapping; and Complete Strong b -metric Space.

1 Introduction and Preliminaries

In 1922, a mathematician Banach [1] proved a very important result regarding a contraction mapping, known as the Banach contraction principle, which states that every self-mapping T defined on a complete metric space (X, d) satisfying

$$\forall x, y \in X, d(Tx, Ty) \leq \lambda d(x, y), \text{ where } \lambda \in (0, 1)$$

has a unique fixed point and for every $x_0 \in X$ a sequence $\{T_n x_0\}_{n=1}^{\infty}$ converges to the fixed point. Subsequently, in 1962, Edelstein [2] proved the following

version of the Banach contraction principle. Let (X, d) be a compact metric space and let $T : X \rightarrow X$ be a self-mapping. Assume that for all $x, y \in X$ with $x \neq y$,

$$d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y).$$

Then T has a unique fixed point in X . In 2012, Wardowski [3] introduced a new type of contractions called F-contraction and proved a new fixed point theorem concerning F-contractions.

Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : R^+ \rightarrow R$ is a mapping satisfying the following conditions:

F1 F is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y$, $F(x) < F(y)$;

F2 For each sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;

F3 There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

We denote by ζ , the set of all functions satisfying the conditions (F1) – (F3). Wardowski [3] then stated a modified version of the Banach contraction principle as follows. Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T_n x\}_{n=1}^\infty$ converges to x^* . In 2014, Hossein, P. and Poom, K. [15] defined the F-Suzuki contraction as follows and gave another version of theorem. Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F-Suzuki-contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$d(x, Tx) < d(x, y) \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : R^+ \rightarrow R$ is a mapping satisfying the following conditions:

F1 F is strictly increasing, i.e. for all $x, y \in R^+$ such that $x < y$, $F(x) < F(y)$;

F2 For each sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;

F3 F is continuous on $(0, \infty)$

We denote by ζ , the set of all functions satisfying the conditions (F1) – (F3).

Let T be a self-mapping of a complete metric space X into itself. Suppose $F \in \zeta$

and there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T_n x_0\}_{n=1}^\infty$ converges to x^* .

Following this direction of research (see examples, [4, 5, 6, 7, 8, 9, 10]), in this paper, fixed point results of Piri and Kumam [11], Ahmad et al [9], Suzuki [16] and Suzuki [17] are extended by introducing common fixed point problem for multivalued generalized F-Suzuki-contraction mappings in strong b-metric spaces.

Definition 1.1 (Hardy and Rogers [14])

- (1) There exist non-negative constants a , satisfying $\sum_{i=1}^5 a_i < 1$ such that, for each $x, y \in X$, $d(f(x), f(y)) < a_1 d(x, y) + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x))$.
- (2) There exist monotonically decreasing functions $a_i(t) : (0, \infty) \rightarrow [0, 1)$ satisfying $\sum_{i=1}^5 a_i(t) < 1$ such that, for each $x, y \in X$, $x \neq y$, $d(f(x), f(y)) < a_1(d(x, y))d(x, f(x)) + a_2(d(x, y))d(y, f(y)) + a_3(d(x, y))d(x, f(y)) + a_4(d(x, y))d(y, f(x)) + a_5(d(x, y))d(x, y)$.
- (3) For each $x, y \in X$, $x \neq y$, $d(f(x), f(y)) < \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}$.

Lemma 1.1 [13] From definition 1.1, (1) $\implies$ (2) $\implies$ (3).

2 Main Results

Definition 2.1 A strong b-metric is a semimetric space (X, d) if there exists $s \geq 1$ for which d satisfies the following triangular inequality.

$$d(x, y) \leq d(x, z) + sd(z, y), \text{ for each } x, y, z \in X. \quad (2.1)$$

Definition 2.2 Let $\mathcal{U}$ be the family of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that:

- (F1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}^+$ such that $x < y$, $F(x) < F(y)$.

$F(y);$

(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty;$

(F3) F is continuous on $(0, \infty)$.

Definition 2.3 Let Ψ be the family of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that ψ is continuous and $\psi(t) = 0$ iff $t = 0$.

Definition 2.4 Let (X, d) be a strong b -metric space. Mappings $T, S : X \rightarrow CB(X)$ are said to be multivalued generalized F -Suzuki-Contraction on (X, d) if there exists $F \in \mathcal{U}$ and $\psi \in \Psi$ such that, $\forall x, y \in X, x \neq y$,

$$\frac{1}{1+s}d(x, Tx) < d(x, y) \text{ and } \frac{1}{1+s}d(y, Sy) < d(y, STx)$$

$\Rightarrow \psi(N_{\phi}(x, y)) + F(s^4 H(Tx, Sy)) \leq F(N_{\phi}(x, y))$ in which

$$\begin{aligned} N_{\phi}(x, y) = & \phi_1(d(x, y))(d(x, y)) + \phi_2(d(x, y))(d(y, STx)) \\ & + \phi_3(d(x, y)) \left(\frac{(d(y, Tx)) + d(x, Sy)}{2s} \right) + \phi_4(d(x, y)) \left(\frac{(d(x, STx)) + d(STx, Sy)}{2s} \right) \\ & + \phi_5(d(x, y))(d(STx, Sy) + d(STx, Tx)) + \phi_6(d(x, y))(d(STx, Sy) + d(Tx, x)) \\ & + \phi_7(d(x, y))(d(Tx, y)) + d(y, Sy) \end{aligned} \quad (2.2)$$

for which $\phi : R^+ \rightarrow [0, 1)$, with $\sum_{i=1}^7 \phi_i(d(x, y)) < 1$, is monotonically decreasing function.

Theorem 2.1 Let (X, d) be a complete strong b -metric space and let $T, S : X \rightarrow CB(X)$ be multivalued generalized F -Suzuki-Contraction mappings. Then T and S has a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_n^{\infty}$ and $\{S^n x\}_n^{\infty}$ converge to x^* .

Proof Let $x_0 = x \in X$. Let $x_{n+1} \in Tx_n$ and $x_{n+2} \in Sx_{n+1} \forall n \in N$. If there exists $n \in N$ such that $d(x_n, Tx_n) = d(x_{n+1}, Sx_{n+1}) = 0$ then $x_{n+1} = x_n = x$ becomes a fixed point of T and S , respectively, therefore the proof is complete. So, suppose that $d(x_n, Tx_n) > 0$ and $d(x_{n+1}, Sx_{n+1}) > 0 \forall n \in N$ then the proof will be divided in to three steps.

Step One we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Let

$$d(x_n, Tx_n) > 0 \text{ and } d(x_{n+1}, Sx_{n+1}) > 0 \ \forall n \in N. \quad (2.3)$$

therefore, we have that

$$\begin{aligned} \frac{1}{s+1}d(x_n, Tx_n) &< d(x_n, Tx_n) \text{ and} \\ \frac{1}{s+1}d(x_{n+1}, Sx_{n+1}) &< d(x_{n+1}, Sx_{n+1}) \ \forall n \in N. \end{aligned} \quad (2.4)$$

From Definition 2.4, we have that

$$F(H(Tx_n, Sx_{n+1})) \leq F(N_\phi(x_n, x_{n+1})) - \psi(N_\phi(x_n, x_{n+1})).$$

Since that

$$\begin{aligned}
& N_\phi(x_n, x_{n+1}) \\
&= \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) \\
&+ \phi_3(d(x_n, x_{n+1})) \left(\frac{d(x_n, x_{n+2})}{2s} \right) + \phi_4(d(x_n, x_{n+1})) \left(\frac{(d(x_n, x_{n+2}))}{2s} \right) \\
&+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\
&+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\
&\leq \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) \\
&+ \phi_3(d(x_n, x_{n+1})) \left(\frac{d(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2})}{2s} \right) \\
&+ \phi_4(d(x_n, x_{n+1})) \left(\frac{d(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2})}{2s} \right) \\
&+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\
&+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\
&\leq \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) \\
&+ \phi_3(d(x_n, x_{n+1})) \left(\frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s} \right) \\
&+ \phi_4(d(x_n, x_{n+1})) \left(\frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s} \right) \\
&+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\
&+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\
&\leq \phi_1(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_2(d(x_n, x_{n+1}))(d(x_{n+1}, x_{n+2})) \\
&+ \phi_3(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_3(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\
&+ \phi_4(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi_4(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \\
&+ \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_6(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) \\
&+ \phi_7(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1}))
\end{aligned}$$

$$\begin{aligned}
&= [\phi_1(d(x_n, x_{n+1})) + \phi_3(d(x_n, x_{n+1})) + \phi_4(d(x_n, x_{n+1})) \\
&\quad + \phi_6(d(x_n, x_{n+1}))](d(x_n, x_{n+1})) \\
&\quad + [\phi_2(d(x_n, x_{n+1})) + \phi_3(d(x_n, x_{n+1})) + \phi_4(d(x_n, x_{n+1})) \\
&\quad + \phi_5(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) + \phi_7(d(x_n, x_{n+1}))](d(x_{n+2}, x_{n+1})) \\
&= \phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) \quad (2.5)
\end{aligned}$$

then from (2.5) and definition 2.4, we get

$$\begin{aligned}
&F(d(x_{n+1}, x_{n+2})) \\
&\leq F(\phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1}))) \\
&\quad - \psi(\phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1}))). \quad (2.6)
\end{aligned}$$

On contrary, if $d(x_{n+1}, x_{n+2}) > d(x_n, x_{n+1})$, then

$$\phi'(d(x_n, x_{n+1}))(d(x_n, x_{n+1})) + \phi''(d(x_n, x_{n+1}))(d(x_{n+2}, x_{n+1})) < d(x_{n+1}, x_{n+2})$$

and therefore (2.6) becomes

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_{n+1}, x_{n+2})) - \psi(d(x_{n+1}, x_{n+2})).$$

But, from (2.3) and the fact that $\psi(d(x_{n+1}, x_{n+2})) > 0$, this is a contradiction. Thus, we conclude that

$$\begin{aligned}
F(d(x_{n+1}, x_{n+2})) &\leq F(d(x_n, x_{n+1})) - \psi(d(x_n, x_{n+1})) \\
&< F(d(x_n, x_{n+1})). \quad (2.7)
\end{aligned}$$

From (2.7) and Definition 2.2(F1), we have that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \quad \forall n \in N. \quad (2.8)$$

Therefore $\{d(x_n, x_{n+1})\}$ is a nonnegative decreasing sequence of real numbers. Thus there exists $\gamma \geq 0$ such that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \gamma$. From (2.7) as $n \rightarrow \infty$, we have that

$$F(\gamma) \leq F(\gamma) - \psi(\gamma).$$

This implies that $\psi(\gamma) = 0$ and thus $\gamma = 0$. Consequently we have that

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (2.9)$$

Now, we claim that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. On contrary, we assume that there exists $\epsilon > 0$ and $n, m \in N$ such that, for all $n \geq n_\epsilon$ and $n_\epsilon < n < m$,

$$d(x_n, x_m) \geq \epsilon \text{ and } d(x_{n-1}, x_m) < \epsilon. \quad (2.10)$$

It implies that

$$\begin{aligned} \epsilon &\leq d(x_n, x_m) \leq d(x_n, x_{n-1}) + sd(x_{n-1}, x_m) \\ &< d(x_n, x_{n-1}) + s\epsilon. \end{aligned} \quad (2.11)$$

From (2.11) and (2.9), we have that

$$\epsilon \leq \limsup_{n \rightarrow \infty} d(x_n, x_m) < s\epsilon. \quad (2.12)$$

By triangle inequality, we have that

$$\begin{aligned} \epsilon &\leq d(x_n, x_m) \leq d(x_n, x_{m+1}) + sd(x_{m+1}, x_m) \\ &\leq d(x_n, x_m) + 2sd(x_{m+1}, x_m). \end{aligned} \quad (2.13)$$

From (2.9), (2.10), (2.12) and (2.13), we have that

$$\epsilon \leq \limsup_{n \rightarrow \infty} d(x_n, x_{m+1}) < s\epsilon. \quad (2.14)$$

Similarly, we have that

$$\begin{aligned} \epsilon &\leq d(x_n, x_m) \leq d(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_m) + (s^2 + 1)d(x_n, x_{n+1}). \end{aligned} \quad (2.15)$$

From (2.9), (2.10), (2.12) and (2.15), we have that

$$\epsilon \leq \limsup_{n \rightarrow \infty} d(v_n, u_{n+1}) < s\epsilon. \quad (2.16)$$

Observe that

$$\begin{aligned} d(x_n, x_{m+1}) &\leq d(x_n, x_{n+1}) + sd(x_{n+1}, x_{m+1}) \\ &\leq d(x_n, x_{n+1}) + s[d(x_{n+1}, x_m) + sd(x_{m+1}, x_m)] \\ &\leq d(x_n, x_{n+1}) + s[d(x_n, x_{n+1}) + sd(x_n, x_m) + sd(x_{m+1}, x_m)]. \end{aligned} \quad (2.17)$$

From (2.17), we have that

$$\frac{\epsilon}{s} \leq \limsup_{n \rightarrow \infty} d(x_{n+1}, x_{m+1}) < s^2 \epsilon. \quad (2.18)$$

From (2.9) and (2.10), we select $n_\epsilon > 0 \in N$ such that

$$\begin{aligned} \frac{1}{s+1} d(x_n, Tx_n) &< \frac{1}{s+1} \epsilon < \epsilon \leq d(x_n, x_m) \quad \forall n \geq n(\epsilon) \\ \Leftrightarrow \frac{1}{s+1} d(x_n, Tx_n) &< \frac{1}{s+1} \epsilon < d(x_n, x_m) \quad \forall n \geq n(\epsilon) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{s+1} d(x_{n+1}, Sx_{n+1}) &< \frac{1}{s+1} \epsilon < \frac{\epsilon}{s} \leq d(x_{n+1}, x_{m+1}) \quad \forall n \geq n_\epsilon \\ \Leftrightarrow \frac{1}{s+1} d(x_{n+1}, Sx_{n+1}) &< \frac{1}{s+1} \epsilon < d(x_{n+1}, x_{m+1}) \quad \forall n \geq n_\epsilon \end{aligned}$$

It follows that from Definition 2.4, we have, for every $n \geq n_\epsilon$

$$F(H(x_{n+1}, x_{m+1})) \leq F(N_\phi(x_n, x_m)) - \psi(N_\phi(x_n, x_m)). \quad (2.19)$$

Since that

$$\begin{aligned} d(x_n, x_m) &\leq N_\phi(x_n, x_m) \\ &= \phi_1(d(x_n, x_m))(d(x_n, x_m)) + \phi_2(d(x_n, x_m))(d(x_{n+2}, x_m)) \\ &+ \phi_3(d(x_n, x_m)) \left(\frac{d(x_{n+1}, x_m) + d(x_n, x_{m+1})}{2s} \right) \\ &+ \phi_4(d(x_n, x_m)) \left(\frac{(d(x_{n+2}, x_n) + d(x_{n+2}, x_{m+1}))}{2s} \right) \\ &+ \phi_5(d(x_n, x_m))(d(x_{n+2}, x_{m+1}) + d(x_{n+2}, x_{n+1})) \\ &+ \phi_6(d(x_n, x_m))(d(x_{n+2}, x_{m+1}) + d(x_n, x_{n+1})) \\ &+ \phi_7(d(x_n, x_m))(d(x_m, x_{n+1}) + d(x_m, x_{m+1})) \\ &\leq \phi_1(d(x_n, x_m))(d(x_n, x_m)) + \phi_2(d(x_n, x_m))(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_m)) \\ &+ \phi_3(d(x_n, x_m)) \left(\frac{d(x_{n+1}, x_m) + d(x_n, x_{m+1})}{2s} \right) \\ &+ \phi_4(d(x_n, x_m)) \left(\frac{(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1})) + sd(x_{n+1}, x_{m+1}))}{2s} \right) \\ &+ \phi_5(d(x_n, x_m))(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_{m+1}) + d(x_{n+2}, x_{n+1})) \\ &+ \phi_6(d(x_n, x_m))(d(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_{m+1}) + d(x_n, x_{n+1})) \\ &+ \phi_7(d(x_n, x_m))(d(x_m, x_{n+1}) + d(x_m, x_{m+1})). \end{aligned} \quad (2.20)$$

From (2.12), (2.14), (2.16), (2.18) and (2.20), we have that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} d(x_n, x_m) &\leq \limsup_{n \rightarrow \infty} N_\phi(x_n, x_m) < \phi_1(\epsilon)(s\epsilon) + \phi_2(\epsilon)(s^2\epsilon) + \phi_3(\epsilon)(\epsilon) \\
&\quad + \phi_4(\epsilon)\left(\frac{s^2\epsilon}{2}\right) + \phi_5(\epsilon)(s^3\epsilon) + \phi_6(\epsilon)(s^3\epsilon) + \phi_7(\epsilon)(s\epsilon) \\
&\leq \max\{s\epsilon, s^2\epsilon, \epsilon, \frac{s\epsilon}{2}, s^3\epsilon, s\epsilon\} \\
&= s^3\epsilon
\end{aligned}$$

and therefore

$$\epsilon \leq \limsup_{n \rightarrow \infty} N_\phi(x_n, x_m) < s^3\epsilon. \quad (2.21)$$

Similarly

$$\epsilon \leq \liminf_{n \rightarrow \infty} N_\phi(x_n, x_m) < s^3\epsilon. \quad (2.22)$$

From (2.19), (2.21) and (2.22), we have that

$$\begin{aligned}
F(s^3\epsilon) &= F\left(s^4 \frac{\epsilon}{s}\right) \leq F\left(s^4 \limsup_{n \rightarrow \infty} H(x_{n+1}, x_{m+1})\right) \\
&\leq F\left(\limsup_{n \rightarrow \infty} N_\phi(x_n, x_m)\right) - \psi\left(\limsup_{n \rightarrow \infty} N_\phi(x_n, x_m)\right) \\
&\leq F(s^3\epsilon) - \psi(\epsilon).
\end{aligned} \quad (2.23)$$

From (2.23) and the fact that $\epsilon > 0$, this is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X . By completeness of X, d , $\{x_n\}_{n=1}^\infty$ and $\{x_{n+1}\}_{n=1}^\infty$ converge to some point $x^* \in X$, that is,

$$\lim_{n \rightarrow \infty} d(x_n, x^*) = 0 \text{ and } \lim_{n \rightarrow \infty} d(x_{n+1}, x^*) = 0. \quad (2.24)$$

If there exists increasing sequences $\{n_k\}, \{n+1_k\} \subset N$ such that $x_{n_k} \in Tx^*$ and $x_{n+1_k} \in Sx^*$ for all $k \in N$. Since Tx^* and Sx^* are closed and

$$\lim_{n \rightarrow \infty} d(x_{n_k}, x^*) = 0 \text{ and } \lim_{n \rightarrow \infty} d(x_{n+1_k}, x^*) = 0,$$

we get $x^* \in Tx^*$ and $x^* \in Sx^*$.

Step Three we show that x^* is a common fixed point of T and S . It suffices to

show that

$$\frac{1}{1+s}d(x_n, Tx_n) < d(x_n, x^*) \text{ and } \frac{1}{1+s}d(x_{n+1}, Sx_{n+1}) < d(x_{n+1}, x^*),$$

for every $n \in N$,

(2.25)

implies

$$F(H(Tx^*, x^*)) \leq F(N_\phi(x^*, Tx^*)) - \psi(N_\phi(x^*, Tx^*))$$

and

$$F(H(Sx^*, x^*)) \leq F(N_\phi(Sx^*, x^*)) - \psi(N_\phi(Sx^*, x^*)),$$

respectively.

On contrary, suppose there exists $m \in N$ such that

$$\frac{1}{1+s}d(x_m, Tx_m) \geq d(x_m, x^*) \text{ or } \frac{1}{1+s}d(x_{m+1}, Sx_{m+1}) \geq d(x_{m+1}, x^*). \quad (2.26)$$

From (2.26), we have that

$$(s+1)d(x_m, x^*) \leq d(x_m, Tx_m) \leq d(x_m, x^*) + sd(Tx_m, x^*)$$

or

$$(s+1)d(x_{m+1}, x^*) \leq d(x_{m+1}, Sx_{m+1}) \leq d(x_{m+1}, x^*) + sd(Sx_{m+1}, x^*),$$

and therefore

$$\begin{aligned} d(x_m, x^*) &\leq d(Tx_m, x^*) = d(x_{m+1}, x^*) \text{ and} \\ d(x_{m+1}, x^*) &\leq d(Sx_{m+1}, x^*) = d(x_{m+2}, x^*). \end{aligned} \quad (2.27)$$

From (2.8), (2.26) and (2.27), this is a contradiction. Hence, (2.25) holds, and therefore

$$\begin{aligned} F(d(x_{n+1}, x^*)) &= F(H(Tx_n, Sx^*)) \\ &\leq F(N_\phi(x_n, x^*)) - \psi(N_\phi(x_n, x^*)), \end{aligned} \quad (2.28)$$

and

$$\begin{aligned} F(d(x_{n+2}, x^*)) &= F(H(Sx_{n+1}, Tx^*)) \\ &\leq F(N_\phi(x_{n+1}, x^*)) - \psi(N_\phi(x_{n+1}, x^*)). \end{aligned} \quad (2.29)$$

Since that

$$\begin{aligned} d(x^*, Tx^*) &\leq N_\phi(x_n, x^*) \\ &= \phi_1(d(x_n, x^*))(d(x_n, x^*)) + \phi_2(d(x_n, x^*))(d(x_{n+2}, x^*)) \\ &\quad + \phi_3(d(x_n, x^*)) \left(\frac{d(x_{n+1}, x^*) + d(x_n, Sx^*)}{2s} \right) \\ &\quad + \phi_4(d(x_n, x^*)) \left(\frac{d(x_n, Sx^*) + d(Sx^*, x_{n+2})}{2s} \right) \\ &\quad + \phi_5(d(x_n, x^*))(d(Sx^*, x_{n+2}) + d(x_{n+1}, Sx^*)) \\ &\quad + \phi_6(d(x_n, x^*))(d(x_n, x_{n+1}) + d(x_{n+2}, Tx^*)) \\ &\quad + \phi_7(d(x_n, x^*))(d(Tx^*, x^*) + d(x^*, x_{n+1})) \\ &\leq \max\{d(x_n, x^*), d(x_{n+2}, x^*), \\ &\quad \frac{d(x_{n+1}, x^*) + d(x_n, Sx^*)}{2s}, \\ &\quad \frac{d(x_n, Sx^*) + sd(Sx^*, x_{n+2}) + d(Sx^*, x_{n+2})}{2s}, \\ &\quad d(Sx^*, x_{n+2}) + d(x_{n+1}, Sx^*), d(x_n, x_{n+1}) + d(x_{n+2}, Tx^*), \\ &\quad d(Tx^*, x^*) + d(x^*, x_{n+1})\}. \end{aligned} \quad (2.30)$$

and

$$\begin{aligned}
d(x^*, Sx^*) &\leq N_\phi(x_{n+1}, x^*) \\
&= \phi_1(d(x_{n+1}, x^*))(d(x_{n+1}, x^*)) + \phi_2(d(x_{n+1}, x^*))(d(x^*, x_{n+3})) \\
&+ \phi_3(x_{n+1}, x^*) \left(\frac{d(x_{n+2}, x^*) + d(x_{n+1}, x^*)}{2s} \right) \\
&+ \phi_4(d(x_{n+1}, x^*)) \left(\frac{d(x_{n+1}, Sx^*) + d(Sx^*, x_{n+3})}{2s} \right) \\
&+ \phi_5(d(x_{n+1}, x^*))(d(x_{n+3}, Sx^*) + d(x_{n+2}, Sx^*)) \\
&+ \phi_6(d(x_{n+1}, x^*))(d(x_{n+1}, x_{n+2}) + d(x_{n+3}, Sx^*)) \\
&+ \phi_7(d(x_{n+1}, x^*))(d(Sx^*, x^*) + d(x^*, x_{n+2})) \\
&\leq \max\{(d(x_{n+1}, x^*), d(x^*, x_{n+3}), \\
&\frac{d(x_{n+2}, x^*) + d(x_{n+1}, x^*)}{2s}, \\
&\frac{d(x_{n+1}, x_{n+2}) + sd(x_{n+2}, Sx^*) + d(Sx^*, x_{n+3})}{2s}, \\
&d(x_{n+3}, Sx^*) + d(x_{n+2}, Sx^*), d(x_{n+1}, x_{n+2}) + d(x_{n+3}, Sx^*), \\
&d(Sx^*, x^*) + d(x^*, x_{n+2})\}. \tag{2.31}
\end{aligned}$$

From (2.24) and (2.30), we have that

$$\lim_{n \rightarrow \infty} N_\phi(x_n, x^*) = d(Tx^*, x^*).$$

From (2.24) and (2.31), we have that

$$\lim_{n \rightarrow \infty} N_\phi(x_{n+1}, x^*) = d(Sx^*, x^*).$$

From (2.28) and (2.29) and by the continuity of F and ψ , we have that

$$F(d(x^*, Tx^*)) \leq F(N_\phi(x^*, Tx^*)) - \psi(N_\phi(x^*, Tx^*)),$$

and

$$F(d(x^*, Sx^*)) \leq F(N_\phi(x^*, Sx^*)) - \psi(N_\phi(x^*, Sx^*)).$$

Hence, since Tx^* and Sx^* are closed then we have that $x^* \in Tx^*$ and $x^* \in Sx^*$, that is, x^* is a fixed point of T and S .

In Theorem 2.1, when $T = S = U$, then we have the following result.

Corollary 2.1.1 *Let (X, d) be a complete strong b -metric space and let $U : X \rightarrow CB(X)$ be a multivalued generalized F -Suzuki-Contraction mapping. Then*

U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^{\infty}$ converges to x^* .

In Corollary 2.1.1, when U is a single-valued then we have another new result as follows.

Corollary 2.1.2 *Let (X, d) be a complete strong b -metric space and let $U : X \rightarrow X$ be a single-valued generalized F -Suzuki-Contraction mapping. Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^{\infty}$ converges to x^* .*

In Theorem 2.1, when T and S are two single-valued then the following result holds.

Corollary 2.1.3 *Let (X, d) be a complete strong b -metric space and let $T, S : X \rightarrow X$ be two single-valued generalized F -Suzuki-Contraction mappings. Then T and S have a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ and $\{S^n x\}_{n=1}^{\infty}$ converge to x^* .*

In Theorem 2.1, when (X, d) is a complete b -metric space then the following new result holds.

Corollary 2.1.4 *Let (X, d) be a complete b -metric space and let $T, S : X \rightarrow X$ be two single-valued generalized F -Suzuki-Contraction mappings. Then T and S have a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ and $\{S^n x\}_{n=1}^{\infty}$ converge to x^* .*

In corollary 2.1.4, when $T = S = U$, then we have the following result.

Corollary 2.1.5 *Let (X, d) be a complete b -metric space and let $U : X \rightarrow CB(X)$ be a multivalued generalized F -Suzuki-Contraction mapping. Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^{\infty}$ converges to x^* .*

Corollary 2.1.6 *Let (X, d) be a complete strong b -metric space and let $U : X \rightarrow CB(X)$ be a multivalued generalized F -Suzuki-Contraction mapping such that there exists $F \in \mathcal{U}$ and $\psi \in \Psi$, $\forall x, y \in X$, $x \neq y$, $\frac{1}{s+1}d(x, Ux) < d(x, y) \Rightarrow$*

$\psi(N(x, y)) + F(s^4 d(Ux, Uy)) \leq F(N(x, y))$ in which

$$\begin{aligned} N(x, y) = & \max\{d(x, y), d(y, U^2x), \\ & \frac{(d(y, Ux)) + d(x, Uy)}{2s}, \frac{(d(x, Uy)) + d(U^2x, Uy)}{2s}, \\ & d(U^2x, Uy) + d(Uy, Ux), d(U^2x, Uy) + d(Ux, x), d(Ux, y)) + d(y, Uy)\}. \end{aligned} \quad (2.32)$$

Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^\infty$ converges to x^* .

Proof from Lemma 1.1, since (2.2) $\Rightarrow$ (2.32) then by the corollary 2.1.1 the result follows immediately.

Corollary 2.1.7 *Let (X, d) be a complete strong b -metric space and let $U : X \rightarrow X$ be a single-valued generalized F -Suzuki-Contraction mapping such that there exists $F \in \mathfrak{U}$ and $\psi \in \Psi$, $\forall x, y \in X$, $x \neq y$, $\frac{1}{s+1}d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^4 d(Ux, Uy)) \leq F(N(x, y))$ in which*

$$\begin{aligned} N(x, y) = & \max\{d(x, y), d(y, U^2x), \\ & \frac{(d(y, Ux)) + d(x, Uy)}{2s}, \frac{(d(x, Uy)) + d(U^2x, Uy)}{2s}, \\ & d(U^2x, Uy) + d(Uy, Ux), d(U^2x, Uy) + d(Ux, x), d(Ux, y)) + d(y, Uy)\}. \end{aligned} \quad (2.33)$$

Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^\infty$ converges to x^* .

Proof from Lemma 1.1, since (2.2) $\Rightarrow$ (2.33) then by the corollary 2.1.2 the result holds.

Corollary 2.1.8 *Let (X, d) be a complete strong b -metric space and let $T, S : X \rightarrow X$ be two single-valued generalized F -Suzuki-Contraction mappings such that there exists $F \in \mathfrak{U}$ and $\psi \in \Psi$, $\forall x, y \in X$, $x \neq y$, $\frac{1}{s+1}d(x, Tx) < d(x, y)$ and $\frac{1}{s+1}d(y, Sx) < d(y, STx) \Rightarrow \psi(N(x, y)) + F(s^4 H(Tx, Sy)) \leq F(N(x, y))$ in*

which

$$\begin{aligned}
N(x, y) = & \max\{d(x, y), d(y, STx), \\
& \frac{(d(y, Tx)) + d(x, Sy)}{2s}, \frac{(d(x, Sy)) + d(STx, Sy)}{2s}, \\
& d(STx, Sy) + d(Sy, Tx), d(STx, Sy) + d(Tx, x), d(Tx, y)) + d(y, Sy)\}. \quad (2.34)
\end{aligned}$$

Then T and S have a common fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^\infty$ and $\{S^n x\}_{n=1}^\infty$ converge to x^* .

Proof from Lemma 1.1, since (2.2) $\Rightarrow$ (2.34) then by the corollary 2.1.4 the result holds.

Corollary 2.1.9 Let (X, d) be a complete b -metric space and let $U : X \rightarrow CB(X)$ be a multivalued generalized F -Suzuki-Contraction mapping such that there exists $F \in \mathfrak{U}$ and $\psi \in \Psi$, $\forall x, y \in X$, $x \neq y$, $\frac{1}{2s}d(x, Ux) < d(x, y) \Rightarrow \psi(N(x, y)) + F(s^6 d(Ux, Uy)) \leq F(N(x, y))$ in which

$$\begin{aligned}
N(x, y) = & \max\{d(x, y), d(y, U^2x), \\
& \frac{(d(y, Ux)) + d(x, Uy)}{2s}, \frac{(d(x, Uy)) + d(U^2x, Uy)}{2s}, \\
& d(U^2x, Uy) + d(Uy, Ux), d(U^2x, Uy) + d(Ux, x), d(Ux, y)) + d(y, Uy)\}. \quad (2.35)
\end{aligned}$$

Then U has a fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{U^n x\}_{n=1}^\infty$ converges to x^* .

Proof from Lemma 1.1, since (2.2) $\Rightarrow$ (2.35) then by the corollary 2.1.5 the result holds.

3 Example

Let $X = [0, 1]$. $T, S : [0, 1] \rightarrow CB([0, 1])$ be defined by $Tx = [0, \frac{x}{2}]$ and $Sy = [0, \frac{y}{2}]$ such that $STx = [0, \frac{x}{8}]$ for all $x \in [0, 1]$. Let d be the usual metric on X . Taking $F(t) = \ln(t) + t$ and $\psi(d(x, y)) = \frac{2}{10^2} |y - \frac{y}{2}|$. Let $x < y$, then $\forall x, y \in [0, 1]$ $d(x, y) > 0$ and $d(y, STx) = |y - \frac{x}{8}| > |y - \frac{y}{8}| = \frac{7}{8}y > \frac{y}{4}$. Now, for $s = 1$, we have that $\frac{1}{2}d(x, Tx) = 0 < d(x, y)$ and $\frac{1}{2}d(y, Sy) = \frac{y}{4} < d(y, STx)$. Without lose of generality, let $d(x, y) = \frac{1}{10^4}$ such that $\phi_n(d(x, y)) = \sqrt[n+1]{d(x, y)}$.

This implies that, $\phi_1(d(x, y)) = \frac{1}{10^2} = \phi_2(d(x, y)) = \phi_6(d(x, y)) = \phi_7(d(x, y))$; $\phi_3(d(x, y)) = \frac{1}{10} = \phi_4(d(x, y)) = \phi_5(d(x, y))$. Therefore, we have that

$$\begin{aligned}
F(H(Tx, Sy)) &= \ln(H(Tx, Sy)) + H(Tx, Sy) \\
&= \ln\left(\left|\frac{y}{2} - \frac{x}{4}\right|\right) + \left|\frac{y}{2} - \frac{x}{4}\right| \\
&\leq \frac{1}{10} \left|\frac{y}{2} - \frac{x}{4}\right| = \frac{1}{10} \left|y - \frac{y}{2} - \frac{x}{4}\right| \\
&\leq \frac{1}{10} \left(\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|\right) \\
&= \frac{1}{10} \left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|}{2}\right) + \frac{1}{10} \left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|}{2}\right) \\
&\leq \frac{1}{10} \left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|}{2}\right) + \frac{1}{10} \left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{x}{8}\right| + \left|\frac{x}{8} - \frac{y}{2}\right|}{2}\right) \\
&= \frac{1}{10} \left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|}{2}\right) + \frac{1}{10} \left(\frac{\left|\frac{x}{8} - \frac{y}{2}\right| + \left|x - \frac{x}{8}\right|}{2}\right) + \frac{1}{10} \left(\left|\frac{y}{2} - \frac{x}{8}\right|\right) \\
&\leq \frac{1}{10} \left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|}{2}\right) + \frac{1}{10} \left(\frac{\left|\frac{x}{8} - \frac{y}{2}\right| + \left|x - \frac{x}{8}\right|}{2}\right) \\
&\quad + \frac{1}{10} \left(\left|\frac{y}{2} - \frac{x}{8}\right| + \left|\frac{x}{8} - \frac{x}{4}\right|\right) \\
&= \frac{1}{10} \left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|}{2}\right) + \frac{1}{10} \left(\frac{\left|\frac{x}{8} - \frac{y}{2}\right| + \left|x - \frac{x}{8}\right|}{2}\right) \\
&\quad + \frac{1}{10} \left(\left|\frac{y}{2} - \frac{x}{8}\right| + \left|\frac{x}{8} - \frac{x}{4}\right|\right) + \frac{1}{10^2} (|x - y|) + \frac{1}{10^2} \left(\left|y - \frac{x}{8}\right|\right) \\
&\quad + \frac{1}{10^2} \left(\left|\frac{x}{8} - \frac{y}{2}\right| + \left|\frac{x}{4} - x\right|\right) + \frac{1}{10^2} \left(\left|\frac{x}{4} - y\right| + \left|y - \frac{y}{2}\right|\right) \\
&\quad - \frac{1}{10^2} \left[(|x - y|) + \left(\left|y - \frac{x}{8}\right|\right) + \left(\left|\frac{x}{8} - \frac{y}{2}\right| + \left|\frac{x}{4} - x\right|\right) + \left(\left|\frac{x}{4} - y\right| + \left|y - \frac{y}{2}\right|\right) \right] \\
&\leq \frac{1}{10} \left(\frac{\left|y - \frac{x}{4}\right| + \left|x - \frac{y}{2}\right|}{2}\right) + \frac{1}{10} \left(\frac{\left|\frac{x}{8} - \frac{y}{2}\right| + \left|x - \frac{x}{8}\right|}{2}\right) \\
&\quad + \frac{1}{10} \left(\left|\frac{y}{2} - \frac{x}{8}\right| + \left|\frac{x}{8} - \frac{x}{4}\right|\right) + \frac{1}{10^2} (|x - y|) + \frac{1}{10^2} \left(\left|y - \frac{x}{8}\right|\right) \\
&\quad + \frac{1}{10^2} \left(\left|\frac{x}{8} - \frac{y}{2}\right| + \left|\frac{x}{4} - x\right|\right) + \frac{1}{10^2} \left(\left|\frac{x}{4} - y\right| + \left|y - \frac{y}{2}\right|\right) - \frac{2}{10^2} \left|y - \frac{y}{2}\right|
\end{aligned}$$

$$\begin{aligned}
&= \phi_1(d(x, y))(d(x, y)) + \phi_2(d(x, y))(d(y, STx)) \\
&+ \phi_3(d(x, y)) \left(\frac{(d(y, Tx)) + d(x, Sy)}{2s} \right) + \phi_4(d(x, y)) \\
&\left(\frac{(d(x, STx)) + d(STx, Sy)}{2s} \right) \\
&+ \phi_5(d(x, y))(d(STx, Sy) + d(STx, Tx)) \\
&+ \phi_6(d(x, y))(d(STx, Sy) + d(Tx, x)) \\
&+ \phi_7(d(x, y))(d(Tx, y)) + d(y, Sy) - \psi(d(x, y))
\end{aligned}$$

4 Conclusion

Fixed point results of Piri and Kumam [11], Ahmad et al [9], Suzuki [16] and Suzuki [17] are extended by introducing common fixed point problem for multivalued generalized F-Suzuki-contraction mappings in strong b-metric spaces. In specific, Corollary 2.1.1 and corollary 2.1.2 generalize and extend the work of Ahmad et al [9] and Kumam and Hossein [5], respectively.

5 Competing interests

The author declares that he has no competing interests.

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References

- [1] Banach, B: *Sur les op rations dans les ensembles abstraits et leur application aux  quations int grales*. Fundam. Math. 3, (1922), 133-181.
- [2] Edelstein, M: *On fixed and periodic points under contractive mappings*. 37,(1962), 74-79.

- [3] Wardowski, D: *Fixed point theory of a new type of contractive mappings in complete metric spaces*. Fixed Point Theory Appl. 2012, Article ID 94 (2012).
- [4] Wardowski, D, Dung, NV: *Fixed points of f -weak contractions on complete metric spaces*. Demonstr. Math. 1, (2014), 146-155.
- [5] Kumam, P, Dung, NV, Sitthithakerngkiet, K: *A generalization of Āł CiriĀłc fixed point theorem*. Filomat 29(7), (2015), 1549-1556.
- [6] Dubey, A.K., Rita, S., Dubey, R.P.: *Some Fixed Point Results in b -metric Spaces*. Asian Journal of Mathematics and Applications. 2014, (2014), 2307-7743.
- [7] Alsulami, H.H., Erdal, K.J., Hossein, P.: *Fixed Points of Generalized F -Suzuki Type Contraction in Complete b -Metric Spaces*. Discrete Dynamics in Nature and Society, 2015, (2015).
- [8] Wardowski, D: *Fixed point theory of a new type of contractive mappings in complete metric spaces*. Fixed Point Theory Appl. 2012, 94, (2012).
- [9] Ahmad et al.: *New fixed point theorems for generalized F -contractions in complete metric spaces*. Fixed Point Theory and Applications. 2015,80, (2015).
- [10] Nguyen, V.D., Vo, T.H.: *A Fixed Point Theorem for Generalized F -Contractions on Complete Metric Spaces*. Vietnam J. Math., (2014).
- [11] Piri, H, Kumam, P: *Some fixed point theorems concerning F -contraction in complete metric spaces*. Fixed Point Theory Appl. 2014, Article ID 210 (2014). doi:10.1186/1687-1812-2014-210.
- [12] Piri, H, Kumam, P: *Fixed point theorems for generalized F -Suzuki-contraction mappings in complete b -metric spaces*. Fixed Point Theory and Applications (2016) 2016:90 DOI 10.1186/s13663-016-0577-5.
- [13] Rhoades, B. E.: *A comparison of various definition of contractive mappings*. Trans. Amer. Math. Soc. 226 (1977) 257 – 290.
- [14] Hardy, G. E., Rogers, T. D.: *A generalization of a fixed point theorem of Reich*. Canad. Math. Bull. 16 (1973), 201-206. MR 48 2847.
- [15] Hossein, P., Kumam, P.: *Some fixed point theorems concerning F -contraction in complete metric spaces*. Fixed point theory and applications, 2014:210, (2014).
- [16] Suzuki, T.: *A generalized Banach contraction principle that characterizes metric completeness*. Proceedings of the American Mathematical Society, 136,5,(2008), 1861-1869.
- [17] Suzuki, T.: *Discussion of several contractions by Jachymski's approach*. Fixed point theory and applications, 2016:91, (2016).